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Element-free solution of geometrically exact rod elastostatics based on intrinsic (material) field variables

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Abstract

Among the one–dimensional models available for non linear analysis of rods, the one proposed by Simó [5] as a extension of the work of Reissner and Antman is capable to treat arbitrarily large rotations of the cross–sections, and is thus called *geometrically exact rod model*. Several general–purpose finite element solutions based on different parmetrizations have been proposed (Simó and Vu-Quoc [6], Ibrahimbegović [2], Jelenić and Crisfield [3] and others). In such solutions the field variables need to be referred to the general spatial frame in order to carry the assembly of the element equations (variables are then said to be expressed in the spatial form). In contrast to this situation, the constitutive equations and the equilibrium equations of the problem are most naturally expressed using intrinsic variables (the material form of the variables). Starting from the late facts, the material form of the variational principles is deduced and element-free solutions based on these principles are presented.

1 Introduction

The *geometrically exact rod model* was formulated by Simo [5] starting from the works of Reissner and Antman [1]. It allows for a exact kinematic description of finite rotations and displacements and has been extensively studied by several authors. Most general-purpose finite element formulations of the model are based on the spatial form of the field variables. This spatial formulation allows for a direct assembly of the element tangent stiffness matrix into the global stiffness, which is very convenient. Although the spatial form of the tangent operator is very concise (if the natural parametrization of rotations is used), it is based on the spatial form of the element constitutive matrix, which has to be evaluated in each integration point by means of transformations of its blocks through the Gauss point rotations, which is a costly operation. Another drawback of the spatial formulation is the need of transforming the section forces to the section reference frame after each increment if such information is needed. In this work we examine the formulation of the model variables and solutions in terms of the intrinsic (material) field variables.

2 Kinematics, deformation variations and field equations

In the following we use Simo's terminology and notation. Intrinsic –also called material– variables (denoted by capital letters) are referred to the section reference frame. Spatial variables (denoted by small letters) are referred to the fixed frame. Cross–sections of the rod are assumed to undergo a rigid body motion, translating and rotating during the deformation process. The position vector of a material point can be written in terms of its relative location into the section $\boldsymbol{r}^*$, and the position of the centroid of the section $\boldsymbol{x}$ as $\boldsymbol{x}^* = \boldsymbol{x} + \boldsymbol{r}^*$ and $\boldsymbol{r}^* = \mathbf{\Lambda}_{\mathrm{d}}\, \mathbf{\Lambda}_0\, \boldsymbol{R}^* = \mathbf{\Lambda}\, \boldsymbol{R}^*$. Section points rotate from a reference (material) configuration (described by $\boldsymbol{R}^*$) to the initial configuration through $\mathbf{\Lambda}_0$, and then to a deformed (actual) configuration through $\mathbf{\Lambda}_{\mathrm{d}}$. Composition of both rotations produces the rotation tensor $\mathbf{\Lambda}$, which together with $\boldsymbol{x}$ are the **configuration functions** of the model. The 1D deformation gradient $\partial \boldsymbol{x}^*/\partial S$ can be written as $\boldsymbol{\Gamma} + \boldsymbol{K} \times \boldsymbol{R}^*$, where $\boldsymbol{\Gamma} = \mathbf{\Lambda}^{\mathsf{T}} \boldsymbol{x}^{'}$ (elongation and shear) and $\widehat{\mathbf{K}} = \mathbf{\Lambda}^{\mathsf{T}} \mathbf{\Lambda}^{'}$ (change of orientation) are the **intrinsic generalized deformations**. The material configuration variations are $\delta\boldsymbol{\chi} = \mathbf{\Lambda}^{\mathsf{T}}\, \delta\boldsymbol{x}$ and $\delta\widehat{\mathbf{\Omega}} = \mathbf{\Lambda}^{\mathsf{T}} \delta\mathbf{\Lambda}$ (the last one is called *spin*). Let's introduce the variations $\delta\boldsymbol{\Gamma}$ and $\delta\boldsymbol{K}$ as conjugate variables to the section forces and moments $\boldsymbol{N}$, $\boldsymbol{M}$. Then, the intrinsic form of the **virtual work equation** (with $\boldsymbol{K}$ and $\delta\boldsymbol{\Omega}$ as axial vectors of the change of orientation and the spin)

$$\int_{\Gamma} (\boldsymbol{N} \cdot \delta\boldsymbol{\Gamma} + \boldsymbol{M} \cdot \delta\boldsymbol{K})\, dS = \int_{\Gamma} (\boldsymbol{Q}_n \cdot \delta\boldsymbol{\chi} + \boldsymbol{Q}_m \cdot \delta\boldsymbol{\Omega})\, dS$$
$$+ \boldsymbol{N}_1 \cdot \delta\boldsymbol{\chi}(S_1) + \boldsymbol{N}_2 \cdot \delta\boldsymbol{\chi}(S_2) + \boldsymbol{M}_1 \cdot \delta\boldsymbol{\Omega}(S_1) + \boldsymbol{M}_2 \cdot \delta\boldsymbol{\Omega}(S_2), \quad (1)$$

holds for every admissible variation $(\delta\boldsymbol{\chi}, \delta\boldsymbol{\Omega})$ iff $(\boldsymbol{x}, \mathbf{\Lambda})$ is an equilibrium configuration. It can be shown from the geometry of the configuration space, that the deformation variations can be expressed as functions of the configuration: $\delta\boldsymbol{\Gamma} = (\delta\boldsymbol{\chi})^{'} + \boldsymbol{K} \times \delta\boldsymbol{\chi} + \boldsymbol{\Gamma} \times \delta\boldsymbol{\Omega}$, and $\delta\boldsymbol{K} = (\delta\boldsymbol{\Omega})^{'} + \boldsymbol{K} \times \delta\boldsymbol{\Omega}$. Integration by parts leads to the **equilibrium equations** $\boldsymbol{N}^{'} + \boldsymbol{K} \times \boldsymbol{N} + \boldsymbol{Q}_n = \mathbf{0}$, and $\boldsymbol{M}^{'} + \boldsymbol{K} \times \boldsymbol{M} + \boldsymbol{\Gamma} \times \boldsymbol{N} + \boldsymbol{Q}_m = \mathbf{0}$. **Consitutitve equations** are naturally established between intrinsic variables.

3 Consistent tangent operator

The consistent linearization of the internal virtual work requires the evaluation of the **second variations** of the configuration, which, using the geometrical properties of the configuration space, lead to the following expressions: $\Delta(\delta\boldsymbol{\chi}) = -\Delta\boldsymbol{\Omega} \times \delta\boldsymbol{\chi}$, $\Delta(\delta\boldsymbol{\Omega}) = \mathbf{0}$, $\Delta(\delta\boldsymbol{\chi}^{'}) = -\Delta\boldsymbol{\Omega}^{'} \times \delta\boldsymbol{\chi} - \Delta\boldsymbol{\Omega} \times \delta\boldsymbol{\chi}^{'}$, $\Delta(\delta\boldsymbol{\Omega}^{'}) = \mathbf{0}$. The linearizaton of the internal virtual work equation leads to the **geometrical** tangent **operator**,

$$\begin{bmatrix} \mathbf{0} & \widehat{\mathbf{K}}\,\widehat{\mathbf{N}} & \mathbf{0} & \mathbf{0} \\ \widehat{\mathbf{N}}\,\widehat{\mathbf{K}} & \widehat{\mathbf{N}}\,\widehat{\mathbf{\Gamma}}+\widehat{\mathbf{M}}\,\widehat{\mathbf{K}} & \widehat{\mathbf{N}} & \widehat{\mathbf{M}} \\ \mathbf{0} & -\widehat{\mathbf{N}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad (2)$$

and to the **constitutive operator**

$$\begin{bmatrix} -\widehat{\mathbf{K}}\mathbf{C}_{\Gamma\Gamma}\widehat{\mathbf{K}} & -\widehat{\mathbf{K}}\,(\mathbf{C}_{\Gamma\Gamma}\widehat{\mathbf{\Gamma}}+\mathbf{C}_{\Gamma K}\widehat{\mathbf{K}}) & -\widehat{\mathbf{K}}\mathbf{C}_{\Gamma\Gamma} & -\widehat{\mathbf{K}}\mathbf{C}_{\Gamma K} \\ -(\widehat{\mathbf{\Gamma}}\mathbf{C}_{\Gamma\Gamma}+\widehat{\mathbf{K}}\mathbf{C}_{K\Gamma})\,\widehat{\mathbf{K}} & -(\widehat{\mathbf{\Gamma}}\mathbf{C}_{\Gamma\Gamma}+\widehat{\mathbf{K}}\mathbf{C}_{K\Gamma})\,\widehat{\mathbf{\Gamma}}-(\widehat{\mathbf{\Gamma}}\mathbf{C}_{\Gamma K}+\widehat{\mathbf{K}}\mathbf{C}_{KK})\,\widehat{\mathbf{K}} & -(\widehat{\mathbf{\Gamma}}\mathbf{C}_{\Gamma\Gamma}+\widehat{\mathbf{K}}\mathbf{C}_{K\Gamma}) & -(\widehat{\mathbf{\Gamma}}\,\mathbf{C}_{\Gamma K}+\widehat{\mathbf{K}}\mathbf{C}_{KK}) \\ \mathbf{C}_{\Gamma\Gamma}\widehat{\mathbf{K}} & \mathbf{C}_{\Gamma\Gamma}\widehat{\mathbf{\Gamma}}+\mathbf{C}_{\Gamma K}\widehat{\mathbf{K}} & \mathbf{C}_{\Gamma\Gamma} & \mathbf{C}_{\Gamma K} \\ \mathbf{C}_{K\Gamma}\widehat{\mathbf{K}} & \mathbf{C}_{K\Gamma}\widehat{\mathbf{\Gamma}}+\mathbf{C}_{KK}\widehat{\mathbf{K}} & \mathbf{C}_{K\Gamma} & \mathbf{C}_{KK} \end{bmatrix}. \quad (3)$$

The virtual work equation can be linearized as $\Delta\,(\delta W_{int} - \delta W_{ext})$. If the configuration $\boldsymbol{x}, \mathbf{\Lambda}$ is an equilibrium configuration for a given load factor λ, then the virtual work is zero for every admissible variation, and $\Delta\delta W_{int} = \Delta\delta W_{ext}$. This expression defines the tangent equilibrium of the geometrically exact model.

4 Numerical experiments

In order to check the performance of the operator we fisrt consider the linear problem consistently derived from the general nonlinear case. For this purpose the generalized deformations may be split into initial and deformational parts in the following manner $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_0 + \boldsymbol{\Gamma}_{\mathrm{d}}$, $\boldsymbol{K} = \boldsymbol{K}_0 + \boldsymbol{K}_{\mathrm{d}}$. In the linear problem, *equilibrium is established in the undeformed configuration*, thus only the initial parts of the deformations appear in the equations. The linear equations are $\boldsymbol{N}^{'} + \boldsymbol{K}_0 \times \boldsymbol{N} + \boldsymbol{Q}_n = \mathbf{0}$ and $\boldsymbol{M}^{'} + \boldsymbol{K}_0 \times \boldsymbol{M} + \boldsymbol{\Gamma}_0 \times \boldsymbol{N} + \boldsymbol{Q}_m = \mathbf{0}$, with $\boldsymbol{\Gamma}_0 = \{1, 0, 0\}$ and $\boldsymbol{K}_0 = \{\tau_0, 0, \kappa_0\}$ (torsion and curvature of the undeformed rod centerline). The initial deformations are not subject to variation, thus only the constitutive part of the tangent operator is derived.

We search numerical solutions using the *Point Interpolation Method* (PIM), Liu [4]. For this purpose N *field nodes* are scattered along the problem domain (the interval $[S_1, S_2]$) in such a way that intervals with high nodal density correspond to high values of curvature and/or torsion. The domain is divided into a certain number of integration cells N_{cells}. For convenience, cell ends are chosen to be coincident with field nodes (although they don't need to). The *support domain* of each cell is defined as an interval with the same center as the integration cell, and radius d_s; the radius is iteratively determined as the minimum value for which the relation between d_s and the support domain nodal average spacing is greater than a pre–determined parameter α_s (typically between 1.0 and 6.0). The number of nodes belonging to the support domain n_s is equal or greater than the number of those belonging to the integration cell. A number of n_s PIM shape functions are constructed using the n_s support domain nodes for each integration cell — this procedure is thoroughly explained in Liu [4]. Integration is carried on by means of a gaussian quadrature; the number of quadrature points in a cell n_g is chosen so that the polynomial shape functions are exactly integrated ($n_g \geq n_s/2$). Shape functions and their derivatives are implemented into the virtual work equation. The solution vector contains the nodal intrinsic configuration increments $\Delta\boldsymbol{\Phi} = \{\Delta\boldsymbol{\chi}, \Delta\boldsymbol{\Omega}\}$, which need to be transformed into spatial form in order to understand them as displacements and rotations referred to the fixed frame $\Delta\boldsymbol{x} = \boldsymbol{\Lambda}_0\,\Delta\boldsymbol{\chi}$ and $\Delta\boldsymbol{\omega} = \boldsymbol{\Lambda}_0\,\Delta\boldsymbol{\Omega}$. Furthermore, it is convenient to work with incremental rotations $\Delta\boldsymbol{\theta}$, which are related to $\Delta\boldsymbol{\omega}$ through $\mathbf{T}(\boldsymbol{\theta})$ (Ibrahimbegović *et al.* [2]). In linear problems it can be shown that $\mathbf{T}(\boldsymbol{\theta}) = \mathbf{1}$, therefore $\Delta\boldsymbol{\theta} = \boldsymbol{\Lambda}_0\,\Delta\boldsymbol{\Omega}$.

A simply supported straight beam with flexural rigidity $EI = 2.5\ 10^2$, shear stiffness $GA = 1.0\ 10^5$, section height $h = 0.1$ and length $L = 10$, under a uniform load $q = 1.0$ is considered. Element–free implementation behavior is tested under different conditions: (a) increasing number of field nodes, ranging from $N = 3$ to $N = 25$, (b) increasing support domain normalized radius, ranging from $\alpha_s = 1.1$ to $\alpha_s = 11.1$; this is equivalent to increasing order of the interpolation polynomials, and (c) varying size of integration cells, ranging from 2– to 7–node cells (1 to 6 intervals). A sensitivity analysis is made for every size of the integration cells. The following deflection–related error parameter is considered for tracing convergence (Liu [4]):

$$err = \frac{\sqrt{\int_0^L (v_{num} - v_{exact})^2\ dS}}{\sqrt{\int_0^L v_{exact}^2\ dS}}. \tag{4}$$

Results show for a given size of the integration cell and for fixed α_s a slow decay of the error parameter with increasing number of field nodes N. If the node number is fixed, the error parameter falls remarkably when the support domain normalized radius α_s reaches a certain value. This value is rather independent of the number of nodes. For 2–node integration cells and $\alpha_s = 4.1$ (interpolation polynomials of order 7, $n_g = 4$), $\log_{10}(err)$ falls under -7. For 3–node integration cells values of $\log_{10}(err)$ under -10 are achieved with $\alpha_s = 3.1$ (interpolation polynomials of order 6, $n_g = 3$). Remarkably, increasing the support domain radius above these values does not lead to lower errors.

In order to reach the same accuracy ($\log_{10}(err)$ under -10), 2–node integration cells need $\alpha_s = 6.1$ (interpolation polynomials of order 11, $n_g = 6$). 4–node cells show, with increasing number of field nodes, a rather unstable behavior for values of α_s under 6.1; interpolation polynomials of order 13 ($\alpha_s = 7.1$, $n_g = 7$) are required to reach high accuracies. 5–node cells show good results with $\alpha_s = 4.1$, and 7–node cells need again $\alpha_s = 7.1$.

Two parameters are directly related to the computational efficiency of the solution: the number of field nodes N and the total number of Gauss points N_g. Focusing our attention on the stable solutions for increasing N, the better choices for different integration cell sizes are shown in the following table. Results show very good convergence properties of mesh–free solutions with a low number of field nodes, 3–node integration cells and interpolation polynomials of order 6. For this case the total number of Gauss points is $N_g = 4N_{cells} - 2$, which is always greater than the number of field nodes $N = 2N_{cells} + 1$. The intrinsic formulation operates with no need of coordinate transformation of the constitutive matrix. Thus, it is very convenient from the computational viewpoint when the number of Gauss points is greater than the number of field nodes, as in the proposed interpolation.

	2–node cell	3–node cell	5–node cell
N_{cells}	12	3	2
N	13	7	9
N_g	60	10	8
α_s (order)	6.1 (7)	3.1 (6)	4.1 (8)

Table 1: Lowest number of field nodes and gauss points necessary to reach high accuracies

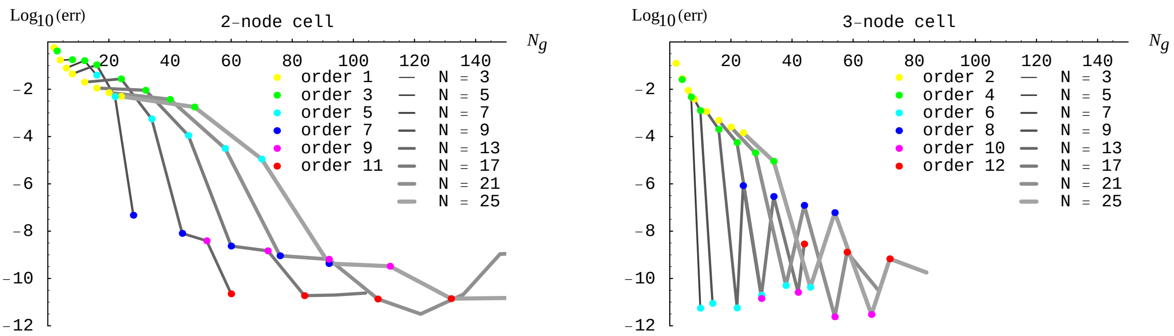

Figure 1: Convergence of the element–free solution

5 Conclusions

In this work, a novel form of the tangent operator for the geometrically exact beam model has been proposed. Geometrical and constitutive parts of the operator are deduced using the geometrical properties of the configuration space. Incremental–iterative solutions can be developed from the tangent operator. The linear version of this procedure has been developed and implemented by means of an element–free method (Point Interpolation Method), which is especially well suited to the 1D problem. In a next stage, nonlinear incremental (–iterative) solutions are being implemented on the basis of the tangent operator introduced in section 3. Economy in the iterative process may be achieved due to the properties of the intrinsic solution.

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