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A new expression of the tangent operator for the geometrically exact beam model

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Abstract

In this work we examine the derivation of the tangent operator for the geometrically exact (Reissner–Simo) beam theory. A new linearization procedure of the virtual work equation, consistent with the co–rotated nature of the generalized deformation variations is proposed. This procedure keeps, as in Simo & Vu–Quoc [5], the position increments and rotation increments (spins) as unknowns, and results in new expressions of the geometric and the load terms of the tangent operator, differing from previous ones in skew-symmetric terms which are equivalent to the moment equilibrium equations and therefore vanish at equilibrium. The performance of our operator is being tested on the basis of a purely incremental analysis.

1 Introduction

Reissner [2] and Simo [4] have developed a beam model with an exact kinematic description of finite rotations and displacements, which is called *geometrically exact beam model*, and has been extensively studied by several authors. This work is concerned with the derivation of a new form of the tangent operator for the Reissner–Simo model.

2 Basic kinematics

Every cross–section of the beam is assumed to undergo a rigid body motion, translating and rotating during the deformation process (fig.1). Therefore, the position vector of a material point can be written in terms of its relative location into the section $\boldsymbol{r}^*$, and the position of the centroid of the section $\boldsymbol{x}$ as

$$\boldsymbol{x}^* = \boldsymbol{x} + \boldsymbol{r}^* \qquad \text{and} \qquad \boldsymbol{r}^* = \boldsymbol{\Lambda}_{\mathrm{d}}\, \boldsymbol{\Lambda}_0\, \boldsymbol{R}^* = \boldsymbol{\Lambda}\, \boldsymbol{R}^*, \tag{1}$$

showing how the section points rotate from a reference (material) configuration (described by $\boldsymbol{R}^*$) to the initial configuration through $\boldsymbol{\Lambda}_0$, and then to a deformed (actual) configuration through $\boldsymbol{\Lambda}_{\mathrm{d}}$. Composition of both rotations produces the rotation tensor $\boldsymbol{\Lambda}$, which together with $\boldsymbol{x}$ are the **configuration functions** of the model. The 1D deformation gradient $\partial \boldsymbol{x}^*/\partial S$ can be written as $\boldsymbol{\gamma} + \widehat{\boldsymbol{\kappa}}\ \boldsymbol{r}^*$, where $\boldsymbol{\gamma} = \boldsymbol{x}^{'}$ and $\widehat{\boldsymbol{\kappa}} = \boldsymbol{\Lambda}^{'}\boldsymbol{\Lambda}^{\mathsf{T}}$ are the **generalized deformations**. Later, the axial vector $\boldsymbol{\kappa}$ of $\widehat{\boldsymbol{\kappa}}$, and the skew–symmetric matrix $\widehat{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$ will appear.

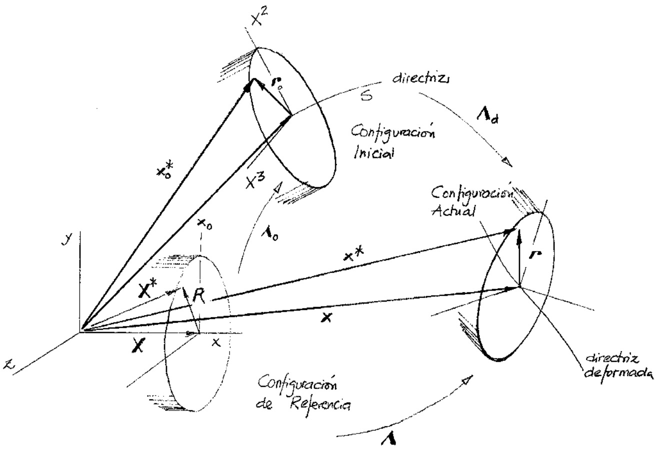

Figure 1: Kinematic description

3 Virtual work equation and co–rotated variations

Evaluation of the variation of a material point position vector gives $\delta\boldsymbol{x}^* = \delta\boldsymbol{x} + \delta\widehat{\boldsymbol{\omega}}\ \boldsymbol{r}^*$, where the new variable $\delta\widehat{\boldsymbol{\omega}} = \delta\boldsymbol{\Lambda}\boldsymbol{\Lambda}^{\mathsf{T}}$ is the *spin* –Ritto–Correa [3]–, which is a skew–symmetric matrix with axial vector $\delta\boldsymbol{\omega}$. Thus $\delta\boldsymbol{x}$ and $\delta\boldsymbol{\omega}$ are the Reissner–Simo model configuration variations.

Following Reissner's procedure in [2], integration by parts of the weighted residual form of the equilibrium equations of the beam $\boldsymbol{n}^{'} + \boldsymbol{q}_n = \boldsymbol{0}$, and $\boldsymbol{m}^{'} + \boldsymbol{\gamma} \times \boldsymbol{n} + \boldsymbol{q}_m = \boldsymbol{0}$, leads to the **virtual work equation** for the Reissner–Simo model

$$\int_{\Gamma} (\boldsymbol{n} \cdot \overset{\nabla}{\delta}\boldsymbol{\gamma} + \boldsymbol{m} \cdot \overset{\nabla}{\delta}\boldsymbol{\kappa})\, dS = \int_{\Gamma} (\boldsymbol{q}_n \cdot \delta\boldsymbol{x} + \boldsymbol{q}_m \cdot \delta\boldsymbol{\omega})\, dS + \boldsymbol{n}_1 \cdot \delta\boldsymbol{x}(S_1) + \boldsymbol{n}_2 \cdot \delta\boldsymbol{x}(S_2) + \boldsymbol{m}_1 \cdot \delta\boldsymbol{\omega}(S_1) + \boldsymbol{m}_2 \cdot \delta\boldsymbol{\omega}(S_2), \quad (2)$$

which states that $(\boldsymbol{x}, \boldsymbol{\Lambda})$ is an equilibrium configuration if (2) holds for every admissible variation $(\delta\boldsymbol{x}, \delta\boldsymbol{\omega})$. The left side represents the virtual work of the internal forces and the variations therein have the following expression

$$\overset{\nabla}{\delta}\boldsymbol{\gamma} = \delta\boldsymbol{x}^{'} + \boldsymbol{\gamma} \times \delta\boldsymbol{\omega} \qquad\qquad \overset{\nabla}{\delta}\boldsymbol{\kappa} = (\delta\boldsymbol{\omega})^{'}. \quad (3)$$

These are the **co–rotated variations** –as in Ritto–Correa [3]– of the generalized deformations. Such kind of variation is defined as the variation from the viewpoint of an observer attached to the section reference frame and can be described as

$$\overset{\nabla}{\delta}\boldsymbol{v} = \boldsymbol{\Lambda}\,\delta(\boldsymbol{\Lambda}^{\mathsf{T}}\boldsymbol{v}) = \delta\boldsymbol{v} - \delta\boldsymbol{\Lambda}\,\boldsymbol{\Lambda}^{\mathsf{T}}\boldsymbol{v} = \delta\boldsymbol{v} - \delta\boldsymbol{\omega} \times \boldsymbol{v}. \quad (4)$$

4 Consistent tangent operator

4.1 Linearization of the internal work

A consistent linearization of the virtual work equation must take into account the properties of the co–rotational variations of the variables, as in the virtual work equations. For this kind of variations, the following derivation rules apply

$$\Delta(\boldsymbol{a}\cdot\boldsymbol{b}) = \overset{\triangledown}{\Delta}\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\overset{\triangledown}{\Delta}\boldsymbol{b} = \Delta\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\Delta\boldsymbol{b}, \tag{5}$$

and

$$\overset{\triangledown}{\Delta}\delta\boldsymbol{x} = -\Delta\widehat{\boldsymbol{\omega}}\,\delta\boldsymbol{x}, \quad \overset{\triangledown}{\Delta}\delta\boldsymbol{\omega} = \boldsymbol{0}, \quad \overset{\triangledown}{\Delta}(\delta\boldsymbol{x}') = -\Delta\widehat{\boldsymbol{\omega}}\,\delta\boldsymbol{x}', \quad \overset{\triangledown}{\Delta}(\delta\boldsymbol{\omega}') = \Delta\widehat{\boldsymbol{\omega}}'\,\delta\boldsymbol{\omega} \tag{6}$$

$$\Delta\delta\boldsymbol{x} = \boldsymbol{0}, \quad \Delta\delta\boldsymbol{\omega} = \Delta\widehat{\boldsymbol{\omega}}\,\delta\boldsymbol{\omega}, \quad \Delta(\delta\boldsymbol{x}') = \boldsymbol{0}, \quad \Delta(\delta\boldsymbol{\omega}') = \Delta\widehat{\boldsymbol{\omega}}'\,\delta\boldsymbol{\omega} + \Delta\widehat{\boldsymbol{\omega}}\,\delta\boldsymbol{\omega}'. \tag{7}$$

Under this rules, the linearization of the internal virtual work integral leads to the geometric and constitutive terms of the tangent operator

$$\begin{aligned}\Delta\delta W_{int} = &\int_\Gamma \{\delta\boldsymbol{x}^\mathsf{T}\ \delta\boldsymbol{\omega}^\mathsf{T}\ \delta\boldsymbol{x}'^\mathsf{T}\ \delta\boldsymbol{\omega}'^\mathsf{T}\} \begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \widehat{\mathbf{n}}\widehat{\boldsymbol{\gamma}} & \widehat{\mathbf{n}} & \widehat{\mathbf{m}}\\ \mathbf{0} & -\widehat{\mathbf{n}} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix} \begin{Bmatrix}\Delta\boldsymbol{x}\\ \Delta\boldsymbol{\omega}\\ \Delta\boldsymbol{x}'\\ \Delta\boldsymbol{\omega}'\end{Bmatrix} dS\\ &+ \int_\Gamma \{\delta\boldsymbol{x}^\mathsf{T}\ \delta\boldsymbol{\omega}^\mathsf{T}\ \delta\boldsymbol{x}'^\mathsf{T}\ \delta\boldsymbol{\omega}'^\mathsf{T}\} \begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & -\widehat{\boldsymbol{\gamma}}\,\mathbf{c}_{\gamma\gamma}\,\widehat{\boldsymbol{\gamma}} & -\widehat{\boldsymbol{\gamma}}\,\mathbf{c}_{\gamma\gamma} & -\widehat{\boldsymbol{\gamma}}\,\mathbf{c}_{\gamma\kappa}\\ \mathbf{0} & \mathbf{c}_{\gamma\gamma}\,\widehat{\boldsymbol{\gamma}} & \mathbf{c}_{\gamma\gamma} & \mathbf{c}_{\gamma\kappa}\\ \mathbf{0} & \mathbf{c}_{\kappa\gamma}\,\widehat{\boldsymbol{\gamma}} & \mathbf{c}_{\kappa\gamma} & \mathbf{c}_{\kappa\kappa}\end{bmatrix} \begin{Bmatrix}\Delta\boldsymbol{x}\\ \Delta\boldsymbol{\omega}\\ \Delta\boldsymbol{x}'\\ \Delta\boldsymbol{\omega}'\end{Bmatrix} dS.\end{aligned} \tag{8}$$

While the constitutive term remains the same as the one first proposed by Simo and Vu–Quoc in [5], the core matrix of the geometric term can be decomposed as

$$\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \widehat{\mathbf{n}}\widehat{\boldsymbol{\gamma}} - \widehat{\boldsymbol{\gamma}}\widehat{\mathbf{n}} & \mathbf{0} & \widehat{\mathbf{m}}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \widehat{\mathbf{m}} & \mathbf{0} & \mathbf{0}\end{bmatrix} + \begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \widehat{\boldsymbol{\gamma}}\widehat{\mathbf{n}} & \widehat{\mathbf{n}} & \mathbf{0}\\ \mathbf{0} & -\widehat{\mathbf{n}} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & -\widehat{\mathbf{m}} & \mathbf{0} & \mathbf{0}\end{bmatrix}. \tag{9}$$

The first (skew-symmetric) matrix appears as a consequence of the non-vanishing second variation of the variations $\Delta\{\delta\boldsymbol{x}^\mathsf{T}\ \delta\boldsymbol{\omega}^\mathsf{T}\ \delta\boldsymbol{x}'^\mathsf{T}\ \delta\boldsymbol{\omega}'^\mathsf{T}\}$. The second matrix is just Simo's operator (our geometric term is, therefore, Simo's transposed).

4.2 Linearization of the external work

The linearization of the virtual work of the external forces using the co–rotated variation leads to

$$\begin{aligned}\Delta\delta W_{int} = &\int_\Gamma \{\delta\boldsymbol{x}^\mathsf{T}\quad \delta\boldsymbol{\omega}^\mathsf{T}\}\ \lambda\Big(\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \widehat{\mathbf{q}}_m\end{bmatrix} + \begin{bmatrix}\mathbf{0} & -\widehat{\mathbf{q}}_n^f\\ \mathbf{0} & -\widehat{\mathbf{q}}_m^f\end{bmatrix}\Big) \begin{Bmatrix}\Delta\boldsymbol{x}\\ \Delta\boldsymbol{\omega}\end{Bmatrix} dS\\ &+ \sum_{a=1}^{2} \{\delta\boldsymbol{x}_a^\mathsf{T}\quad \delta\boldsymbol{\omega}_a^\mathsf{T}\}\ \lambda\Big(\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \widehat{\mathbf{m}}_a\end{bmatrix} + \begin{bmatrix}\mathbf{0} & -\widehat{\mathbf{n}}_a^f\\ \mathbf{0} & -\widehat{\mathbf{m}}_a^f\end{bmatrix}\Big) \begin{Bmatrix}\Delta\boldsymbol{x}_a\\ \Delta\boldsymbol{\omega}_a\end{Bmatrix}\\ &+ \Big(\int_\Gamma \{\delta\boldsymbol{x}^\mathsf{T}\quad \delta\boldsymbol{\omega}^\mathsf{T}\} \begin{Bmatrix}\boldsymbol{q}_n\\ \boldsymbol{q}_m\end{Bmatrix} dS + \sum_{a=1}^{2} \{\delta\boldsymbol{x}_a^\mathsf{T}\quad \delta\boldsymbol{\omega}_a^\mathsf{T}\} \begin{Bmatrix}\boldsymbol{n}_a\\ \boldsymbol{m}_a\end{Bmatrix}\Big)\ \Delta\lambda,\end{aligned} \tag{10}$$

where superscript f stands for the follower (intrinsic) part of the outer forces. This result is also in sharp contrast with Simo's: the core terms are also the sum of a skew-symmetric matrix (with the distributed moments or the end moments) plus the same terms obtained by Simo (with the follower parts of the outer forces).

4.3 Tangent operator

The consistent linearization of the whole virtual work equation can be expressed as $\Delta\,(\delta W_{int} - \delta W_{ext})$. If the configuration $\boldsymbol{x}, \boldsymbol{\Lambda}$ is an equilibrium configuration for a given load factor λ, then the virtual work is zero for every admissible variation, and $\Delta\delta W_{int} = \Delta\delta W_{ext}$. This expression defines the tangent equilibrium of the Reissner–Simo model. Using $\delta\boldsymbol{\phi}^{\mathsf{T}} = \{\delta\boldsymbol{x}^{\mathsf{T}}\ \ \delta\boldsymbol{\omega}^{\mathsf{T}}\}$, $\boldsymbol{q}^{\mathsf{T}} = \{\boldsymbol{q}_n^{\mathsf{T}}\ \ \boldsymbol{q}_m^{\mathsf{T}}\}$ and $\boldsymbol{f}_i^{\mathsf{T}} = \{\boldsymbol{n}_i^{\mathsf{T}}\ \ \boldsymbol{m}_i^{\mathsf{T}}\}$, it can be written as

$$\int_{\Gamma} \{\delta\boldsymbol{\phi}^{\mathsf{T}}\ \delta\boldsymbol{\phi}'^{\mathsf{T}}\}\, \mathbf{d} \left\{ \begin{matrix} \Delta\boldsymbol{\phi} \\ \Delta\boldsymbol{\phi}' \end{matrix} \right\} dS - \sum_{a=1}^{2} \delta\boldsymbol{\phi}_a^{\mathsf{T}}\, \lambda\, \mathbf{f}_a^s\, \Delta\boldsymbol{\phi}_a = \Big(\int_{\Gamma} \delta\boldsymbol{\phi}^{\mathsf{T}}\, \boldsymbol{q}\, dS + \delta\boldsymbol{\phi}_1^{\mathsf{T}}\, \boldsymbol{f}_1 + \delta\boldsymbol{\phi}_2^{\mathsf{T}}\, \boldsymbol{f}_2 \Big) \Delta\lambda, \tag{11}$$

with $\mathbf{d} = \mathbf{d}^M + \mathbf{d}^G - \lambda\ \mathbf{d}^Q$, and each term stands for the constitutive, the geometric and the load core matrix, respectively. A very interesting feature of our operator is that, at equilibrium, the sum of the skew–symmetric terms is (after integration by parts) equivalent to the moment internal equilibrium equation and to the end equilibrium equations, and therefore vanish. This property is **not** restricted to conservative loading –as in [5, remark 4.1]–. Thus, the evaluation of our tangent operator at a point on the equilibrium path should give the same result as with Simo's expressions.

5 Evaluation

A proper discretization of equation (11) leads to the incremental equation $\mathbf{K}^{\nabla}\ \Delta\boldsymbol{\phi} = \mathbf{f}\ \Delta\lambda$, which is the basis of purely incremental solution procedures. Update of the nodal rotations takes into account the geometrical structure of the model. Update of generalized deformations is carried on following the procedures proposed by Ibrahimbegović & Taylor [1]. In this kind of solutions, the predicted force-displacement path drifts away form the equilibrium path, and the use of our operator can be advantageous. The performance of our operator for purely incremental solution procedures is currently being tested on some examples.

6 Conclusions

In this work, a new form of the tangent operator for the Reissner–Simo beam model has been proposed. It is derived form a consistent linearization of the virtual work equation using the properties of co–rotated variations. This process leads to new expressions for the geometric and the load terms of the operator, differing from Simo's in skew-symmetric terms which are equivalent to the moment equilibrium equations and vanish at equilibrium. Numerical performance for purely incremental solution procedures is now under test.

References

[1] Ibrahimbegović A, Taylor RL. On the role of frame–invariance in structural mechanics models at finite rotations, *Computer Methods in Applied Mechanics and Engineering* 2002; **191**: 5159–5176.

[2] Reissner E. On finite deformations of space–curved beams, *Journal of Applied Mathematics and Physics (ZAMP)* 1981; **32**: 734–744.

[3] Ritto-Correa M, Camotim D. On the differentiation of the Rodrigues formula and its significance for the vector–like parametrization of Reissner–Simo beam theory, *International Journal for Numerical Methods in Engineering* 2002; **55**: 1005–1032.

[4] Simo JC, A finite strain beam formulation. The three–dimensional dynamic problem. Part I, *Computer Methods in Applied Mechanics and Engineering* 1985; **49**: 55–70, North–Holland.

[5] Simo JC, Vu-Quoc L. A three–dimensional finite–strain rod model. Part II: Computational aspects, *Computer Methods in Applied Mechanics and Engineering* 1986; **58**: 79–116, North–Holland.